

Parameter estimation for controlled partially observed fractional diffusion process

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Abstract

We consider a controlled second order differential equation which is partially observed with an additional fractional noise. we study the asymptotic (for large observation time) design problem of the input and give an efficient estimator of the unknown signal drift parameter. When the input depends on the unknown parameter, we will try the two-step estimation procedure using the Newton-Raphon method.

1 Introduction

1.1 Historical survey

Over the last decades the experiment design has been given a great deal of interest from the early statistics literature (see *e.g.* [13, 22, 23]) as well as in the engineering literature (see *e.g.* [8, 9, 10]).

The main goal for the problem of the experiment design consists on a two-step procedure: find the energy constraint of the input which can maximize the Fisher information and fix an adaptive estimator under this input. In this area, there are several approaches like sequential design and Bayesian design (see *e.g.* [10, 16, 19] and the references therein).

We will also find some works which concern on the partially observed models such as [1, 17, 19, 20, 21], where linear signal - observation model perturbed by the white noise has been considered.

On the other hand, large sample asymptotic properties (the consistency and the asymptotical normality) of the Maximum Likelihood Estimator (MLE) with the fractional noise [6, 14, 2, 3, 7] have been got enough attention.

Some models of the experiment design with the fractional noise have been studied by Brouste, Cai, Kleptsyna and Popier [24, 4, 5]. However, all these works have one thing in common: the optimal input does not depend the unknow parameters, so it is very easy to obtain directly the Maximum Likelihood Estimator. In this paper, even some technical methods will be the same of the previous works, but the calculation is more complicated and the most important we will deal with the problem that: the input will depend on the parameter. A two-step estimation will be presented.

The paper falls into four parts. In this introduction, we state our models and then we will give our main results in the second part. In the third part, we will try to do some transformation of the models and present the Newton-Raphon method. The proofs of two lemmas will be end all our works.

1.2 The Model And Statement Of The Problem

We consider real-valued functions $x(t)$, $u(t)$, $t \geq 0$ and a process $Y = (Y_t, t \geq 0)$, representing the signal and the observation respectively, governed by the following homogeneous linear system of ordinary and stochastic differential equations interpreted as integral equations:

$$\begin{cases} \frac{d^2x}{dt^2} + k\frac{dx}{dt} + \vartheta x &= u(t), \quad x(0) = 0, \\ dY_t &= x(t)dt + dV_t^H, \quad Y_0 = 0. \end{cases} \quad (1)$$

Here, $V^H = (V_t^H, t \geq 0)$ is normalized fBm with Hurst Index $H \in [\frac{1}{2}, 1)$ and the coefficient ϑ and k are positive constants. System (1) has a uniquely defined solution process (x, Y) where Y is Gaussian but neither Markovian nor a semi martingale for $H \neq \frac{1}{2}$.

Suppose that the parameter ϑ is unknown and is to be estimated given the observed trajectory $Y^T = (Y_t, 0 \leq t \leq T)$.

or a fixed value of the parameter ϑ , let \mathbf{P}_ϑ^T denote the probability measure, induced by (X^T, Y^T) on the function space $\mathcal{C}_{[0,T]} \times \mathcal{C}_{[0,T]}$ and let \mathcal{F}_t^Y be the natural filtration of Y , $\mathcal{F}_t^Y = \sigma(Y_s, 0 \leq s \leq t)$.

Let $\mathcal{L}(\vartheta, Y^T)$ be the likelihood, *i.e.* the Radon-Nikodym derivative of \mathbf{P}_ϑ^T , restricted to \mathcal{F}_T^Y with respect to some reference measure on $\mathcal{C}_{[0,T]}$. In this

setting, Fisher information stands for :

$$\mathcal{I}_T(\vartheta, u) = -\mathbf{E}_{\vartheta} \frac{\partial^2}{\partial \vartheta^2} \ln \mathcal{L}_T(\vartheta, Y^T).$$

Let us denote \mathcal{U}_T some functional space of controls, that is defined by equation (11) and (10). Let us therefore note

$$\mathcal{J}_T(\vartheta) = \sup_{u \in \mathcal{U}_T} \mathcal{I}_T(\vartheta, u)$$

Our main goal is to find estimator $\bar{\vartheta}_T$ of the parameter ϑ which are asymptotically efficient in the sense that, for any compact $\mathbb{K} \subset \mathbb{R}_*^+$,

$$\sup_{\vartheta \in \mathbb{K}} \mathcal{J}_T(\vartheta) \mathbf{E}_{\vartheta} (\bar{\vartheta}_T - \vartheta)^2 = 1 + o(1), \quad (2)$$

as $T \rightarrow \infty$.

2 Main Result

In this section, we will divide two different cases, we will get the optimal input and study the Properties of the MLE.

2.1 Case of $k^2 \geq 2\vartheta$

In this subsection we will consider only the case that $k^2 \geq 2\vartheta$

Theorem 1. *The asymptotical optimal input in the class of controls is \mathcal{U}_T is $u_{opt}^1(t) = \frac{\kappa_H}{\sqrt{2\lambda}} t^{H-\frac{1}{2}}$ where*

$$\kappa_H = 2H\Gamma\left(\frac{3}{2} - H\right)\Gamma\left(\frac{1}{2} + H\right) \quad \text{and} \quad \lambda = \frac{H\Gamma(3-2H)\Gamma(H+\frac{1}{2})}{2(1-H)\Gamma(\frac{3}{2}-H)} \quad (3)$$

and Γ stands for the Gamma function. Moreover,

$$\lim_{T \rightarrow +\infty} \frac{\mathcal{J}_T^1(\vartheta)}{T} = \mathcal{I}^1(\vartheta)$$

where

$$\mathcal{I}^1(\vartheta) = \frac{1}{\vartheta^4} \quad (4)$$

We denote here the MLE $\hat{\vartheta}_T^1$, As the optimal input does not depend on ϑ , the MLE reaches efficiency and we deduce its large asymptotic properties:

Theorem 2. *The MLE is uniformly consistent on compacts $\mathbb{K} \subset \mathbb{R}_*^+$, i.e. for any $\nu > 0$,*

$$\lim_{T \rightarrow \infty} \sup_{\vartheta \in \mathbb{K}} \mathbf{P}_{\vartheta}^T \left\{ \left| \hat{\vartheta}_T^1 - \vartheta \right| > \nu \right\} = 0,$$

uniformly on compacts asymptotically normal: as T tends to $+\infty$,

$$\lim_{T \rightarrow \infty} \sup_{\vartheta \in \mathbb{K}} \left| \mathbf{E}_{\vartheta} f \left(\sqrt{T} \left(\hat{\vartheta}_T^1 - \vartheta \right) \right) - \mathbf{E} f(\xi) \right| = 0 \quad \forall f \in \mathcal{C}_b$$

*and ξ is a zero mean Gaussian random variable of variance $(\mathcal{I}^1(\vartheta))^{-1}$ (see (4) for the explicit value) **which does not depend on H** and we have the uniform on $\vartheta \in \mathbb{K}$ convergence of the moments: for any $p > 0$,*

$$\lim_{T \rightarrow \infty} \sup_{\vartheta \in \mathbb{K}} \left| \mathbf{E}_{\vartheta} \left| \sqrt{T} \left(\hat{\vartheta}_T^1 - \vartheta \right) \right|^p - \mathbf{E} |\xi|^p \right| = 0.$$

Finally, the MLE is efficient in the sense of (2).

2.2 Case of $k^2 < 2\vartheta$

In this section, we consider only when $k^2 < 2\vartheta$. First of all, we will get the optimal input:

Theorem 3. *The asymptotical optimal input in the class of controls \mathcal{U}_T is $u_{opt}^2(t) = \frac{\kappa_H}{\sqrt{2\lambda}} t^{H-\frac{1}{2}} e^{i\omega t}$ where $\omega = \pm \sqrt{\vartheta - \frac{k^2}{2}}$. Moreover,*

$$\lim_{T \rightarrow +\infty} \frac{\mathcal{J}_T^2(\vartheta)}{T} = \mathcal{I}^2(\vartheta)$$

where

$$\mathcal{I}^2(\vartheta) = \frac{16}{(k^4 - 4k^2\vartheta)^2} \quad (5)$$

In this case, the optimal input depends on the parameter ϑ , we can not directly study the properties of MLE, we will use Newton-Raphon method to get the asymptotical properties of MLE which will considered in the Next section.

3 Preliminary Results

3.1 Transformation of The Model

The explicit representation of the likelihood function can be written thanks to the transformation of observation model proposed in [15]. In what follows, all random variables and processes are defined on a given stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbf{P})$ satisfying the usual conditions and processes are (\mathcal{F}_t) -adapted. More over the *natural filtration* of a process is understood as the \mathbf{P} -completion of the filtration generated by this process. Let us define:

$$k_H(t, s) = \kappa_H^{-1} s^{\frac{1}{2}-H} (t-s)^{\frac{1}{2}-H}, \quad w_H(t) = \frac{1}{2\lambda(2-2H)} t^{2-2H},$$

$$N_t = \int_0^t k_H(t, s) dV_s^H$$

where κ_H and λ are defined in (3). Then the process $N = (N_t, t \geq 0)$ is a Gaussian martingale, called in [12] the *fundamental martingale*, whose variance function is noting but w_H . More over, the natural filtration of the martingale N coincides with the natural filtration of the fBm V^H .

Following [15], let us introduce a process $Z = (Z_t, 0 \leq t \leq T)$ the fundamental semimartingale associated to Y , defined as

$$Z_t = \int_0^t k_H(t, s) dY_s. \quad (6)$$

Note that Y can be represented as $Y_t = \int_0^t K_H(t, s) dZ_s$, where $K_H(t, s) = H(2H-1) \int_s^t r^{H-\frac{1}{2}} (r-s)^{H-\frac{3}{2}} dr$ for $0 \leq s \leq t$ and therefore the natural filtration of Y and Z coincide. Moreover, we have the following representation:

$$dZ_t = \lambda \ell(t)^* \zeta(t) d\langle N \rangle_t + dN_t, \quad Z_0 = 0 \quad (7)$$

where $\zeta(t)$ is the solution of the ordinary differential equation

$$\frac{d\zeta(t)}{d\langle N \rangle_t} = \lambda A_0 \otimes \mathbf{A}(t) \zeta(t) + b(t) v(t), \quad \zeta(0) = 0. \quad (8)$$

with

$$\ell(t) = \begin{pmatrix} t^{2H-1} \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad A_0 = \begin{pmatrix} 0 & 1 \\ -\vartheta & -k \end{pmatrix}, \quad \mathbf{A}(t) = \begin{pmatrix} t^{2H-1} & 1 \\ t^{4H-2} & t^{2H-1} \end{pmatrix}, \quad b(t) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ t^{2H-1} \end{pmatrix} \quad (9)$$

Here, for a control $u(t)$ we define the function $v(t)$ by the following equation

$$v(t) = \frac{d}{dw_H(t)} \int_0^t k_H(t, s) u(s) ds; \quad (10)$$

provided that the fractional derivative exists. Let us define the space of control for $v(t)$ that:

$$\mathcal{V}_T = \left\{ v \mid \frac{1}{T} \int_0^T |v(t)|^2 dw_H(t) \leq 1 \right\}. \quad (11)$$

Note that these sets are non empty. Remark that with (10) the following relation between control u and its transformation v holds:

$$u(t) = \frac{d}{dt} \int_0^t K_H(t, s) v(s) dw_H(s). \quad (12)$$

3.2 Likelihood function and the Fisher information

The classical Girsanov theorem gives the following equality

$$\mathcal{L}(\vartheta, Z^T) = \exp \left\{ \lambda \int_0^T \ell(t)^* \zeta(t) dZ_t - \frac{\lambda^2}{2} \int_0^T \zeta(t)^* \ell(t) \ell(t)^* \zeta(t) d\langle N \rangle_t \right\} \quad (13)$$

The fisher information stands for:

$$\mathcal{I}_T(\vartheta, v) = -\mathbf{E}_\vartheta \frac{\partial^2}{\partial \vartheta^2} \ln \mathcal{L}(\vartheta, Z^T)$$

which is

$$\mathcal{I}_T(\vartheta, v) = \lambda^2 \int_0^T \left(\frac{\partial \zeta(t)}{\partial \vartheta} \right)^* \ell(t) \ell(t)^* \frac{\partial \zeta(t)}{\partial \vartheta} d\langle N \rangle_t. \quad (14)$$

3.3 Proof of Theorem 1 and 3

Let us define

$$\mathcal{J}(\vartheta) = \sup_{v \in \mathcal{V}_T} \mathcal{I}_T(\vartheta, v)$$

From (8) we get

$$\zeta(t) = \varphi(t) \int_0^t \varphi^{-1}(s) b(s) v(s) d\langle N \rangle_s \quad (15)$$

where $\varphi(t)$ is the fundamental matrix satisfying:

$$\frac{d\varphi(t)}{d\langle N \rangle_t} = \lambda A_0 \otimes \mathbf{A}(t)\varphi(t), \quad \varphi(t) = \mathbf{Id}, \quad (16)$$

where \mathbf{Id} is the 4×4 identity matrix. Therefore

$$\mathcal{I}_T(\vartheta, v) = \int_0^T \int_0^T K_T(s, \sigma) \frac{s^{\frac{1}{2}-H}}{\sqrt{2\lambda}} v(s) \frac{\sigma^{\frac{1}{2}-H}}{\sqrt{2\lambda}} v(\sigma) ds d\sigma, \quad (17)$$

where

$$K_T(s, \sigma) = \int_{\max(s, \sigma)}^T G(t, s) G(t, \sigma) dt, \quad (18)$$

and

$$G(t, \sigma) = \frac{\partial}{\partial \vartheta} \left(\frac{1}{2} t^{\frac{1}{2}-H} \ell(t)^* \varphi(t) \varphi^{-1}(\sigma) b(\sigma) \sigma^{\frac{1}{2}-H} \right). \quad (19)$$

Then

$$\begin{aligned} \mathcal{J}_T(\vartheta) &= T \sup_{\tilde{v} \in L^2[0, T], \|\tilde{v}\| \leq 1} \int_0^T \int_0^T K_T(s, \sigma) \tilde{v}(s) \tilde{v}(\sigma) ds d\sigma, \\ &= T \sup_{\tilde{v} \in L^2[0, T], \|\tilde{v}\| \leq 1} (K_T \tilde{v}, \tilde{v}) \end{aligned}$$

where $\tilde{v}(s) = \frac{s^{\frac{1}{2}-H}}{\sqrt{2\lambda}} \frac{v(s)}{\sqrt{T}}$ and $\|\cdot\|$ stands for the usual norm in $L^2[0, T]$. So in order to prove the Theorem 1 and 3, we need two Lemma.

Lemma 3.1. *When $k^2 \geq 2\vartheta$,*

$$\lim_{T \rightarrow \infty} \sup_{\tilde{v} \in L^2[0, T], \|\tilde{v}\| \leq 1} (K_T \tilde{v}, \tilde{v}) = \frac{1}{\vartheta^4}$$

with an optimal input $v_{opt}^1(t) = \sqrt{2\lambda} t^{H-\frac{1}{2}}$ belonging to the space of control \mathcal{V}_T .

Lemma 3.2. *When $k^2 < 2\vartheta$,*

$$\lim_{T \rightarrow \infty} \sup_{\tilde{v} \in L^2[0, T], \|\tilde{v}\| \leq 1} (K_T \tilde{v}, \tilde{v}) = \frac{16}{(k^4 - 4k^2\vartheta)^2}$$

with the optimal input $v_{opt}^2(t) = \sqrt{2\lambda} t^{H-\frac{1}{2}} e^{i\omega t}$ where $\omega = \pm \sqrt{\vartheta - \frac{k^2}{2}}$.

The proof of Lemma 3.1 and 3.2 are based on the Laplace transformation computation and will presented in Section 4.

3.4 Proof Of Theorem 2

In order to prove the Theorem 2, we need to check the Ibragimov-Khasminskii Theorem about the asymptotic efficiency for the MLE in [11].

3.4.1 Ibragimov-Khasminskii Theorem

Theorem 4. *Assume that we are given an observable process*

$$d\eta_t = X_T(\vartheta, t)d\langle N \rangle_t + dN_t, \quad t \in [0, T], \quad \vartheta \in \mathbb{K}$$

with the following conditions holds:

- 1.) *The Fisher information $\mathcal{I}_T(\vartheta) \rightarrow \infty$ as $T \rightarrow \infty$ (or $T \rightarrow 0$) uniformly with respect to $\vartheta \in \mathbb{K}$.*
- 2.) *The ratio $\mathcal{I}_T(\vartheta_1)/\mathcal{I}_T(\vartheta_2)$ is uniformly (with respect to T and ϑ_i) bounded.*
- 3.) *The function $\vartheta \rightarrow X_T(\vartheta, t)$ is continuously differentiable.*
- 4.) *The function $F(h) = \int_0^T |X_T(\vartheta + h\mathcal{I}_T(\vartheta)^{-\frac{1}{2}}, t) - X_T(\vartheta, t)|^2 d\langle N \rangle_t$ is greater than $C \min(|h|^2, |h|^\beta)$ for $\vartheta, \vartheta + h\mathcal{I}_T(\vartheta)^{-\frac{1}{2}} \in \mathbb{K}$, where C and β are positive constants.*

Then $\mathcal{I}_T^{1/2}(\vartheta)(\hat{\vartheta}_T - \vartheta) \Rightarrow \mathcal{N}(0, 1)$ as $T \rightarrow \infty$ (or $T \rightarrow 0$), where $\hat{\vartheta}_T$ is the maximum likelihood estimator for ϑ . Moreover, all moments of $\mathcal{I}_T^{1/2}(\vartheta)(\hat{\vartheta}_T - \vartheta)$ tend to the corresponding moments of $\mathcal{N}(0, 1)$. The convergence is uniform with respect to $\vartheta \in \mathbb{K}$.

3.4.2 Taylor's Development Proof

When $k^2 \geq 2\vartheta$, with the optimal input, we can get the new system

$$\begin{cases} \frac{d\zeta^1(t)}{d\langle N \rangle_t} &= \lambda A_0 \otimes \mathbf{A}(t)\zeta^1(t) + b(t)v_{opt}^1(t), \quad \zeta^1(0) = 0, \\ dZ_t^1 &= \lambda \ell(t)^* \zeta^1(t) d\langle N \rangle_t + dN_t, \quad Z_0^1 = 0. \end{cases} \quad (20)$$

Let us define the function

$$g(\vartheta, t) = t^{\frac{1}{2}} \ell(t)^* \varphi(\vartheta, t) \int_0^t \varphi^{-1}(\vartheta, s) b(s) s^{\frac{1}{2}-H} ds \quad (21)$$

where

$$\frac{d\varphi(\vartheta, t)}{dt} = \frac{A_0(\vartheta)}{2} \otimes \mathbf{A}_H(t) \varphi(\vartheta, t).$$

and

$$\mathbf{A}_H(t) = \begin{pmatrix} 1 & t^{1-2H} \\ t^{2H-1} & 1 \end{pmatrix}.$$

With Taylor's development with respect to t , we can get that

$$|g(\vartheta + h, t) - g(\vartheta, t)| = C|h|t^4 + o(t^4) \quad (22)$$

for every real value h , C is a constant which does not depend on ϑ . Here $o(t^4)$ when $t \rightarrow 0$. In our case, the Fisher Information

$$\begin{aligned} \mathcal{I}_T^1(\vartheta, v_{opt}^1) &= \lambda^2 \int_0^T \left(\ell(t)^* \frac{\partial \zeta^1(t)}{\partial \vartheta} \right)^2 d\langle N \rangle_t \\ &= \frac{1}{4} \int_0^T \left| \frac{\partial g(\vartheta, t)}{\partial \vartheta} \right|^2 dt. \end{aligned}$$

with the condition (22), we can verify the four conditions in Theorem 4. So that we can get that $\sqrt{T}(\hat{\vartheta}_T^1 - \vartheta) \Rightarrow \mathcal{N}(0, (\mathcal{I}^1(\vartheta))^{-1})$, and moreover, we can get all of the results in the theorem 2.

3.5 Asymptotical Properties of MLE When $k^2 < 2\vartheta$

When $k^2 < 2\vartheta$, with Lemma 3.2 we know that $v_{opt}^2(t) = \sqrt{2\lambda}t^{H-\frac{1}{2}}e^{i\omega t}$ where $\omega = \pm\sqrt{\vartheta - \frac{k^2}{2}}$. The optimal input depends on the unknown parameter ϑ . So we can not directly use the Ibragimov-Khasminskii Theorem to find the asymptotic properties of MLE. We follow the general procedure : Divide the observation time interval into two parts, the first one being relatively short. Then find a preliminary estimate $\bar{\vartheta}$ of the unknown parameter ϑ from the observation in this interval by using an input which does not depend on ϑ . After that, we use $\bar{\vartheta}$, instead of ϑ , to form an approximately optimal input in the second(long) interval. By using this input, we arrive at an asymptotically efficient estimator of ϑ . At the second stage, we can also use the MLE, though this is not an easily-implemented procedure. A simpler method of the Newton-Raphson type can be described as follows.

3.5.1 Newton-Raphson method

When $k^2 < 2\vartheta$, our system is that

$$\begin{cases} \frac{d\zeta^2(t)}{d\langle N \rangle_t} = \lambda A_0 \otimes \mathbf{A}(t)\zeta^2(t) + b(t)v_{opt}^2(t), & \zeta^2(0) = 0, \\ dZ_t^2 = \lambda \ell(t)^* \zeta^2(t) d\langle N \rangle_t + dN_t, & Z_0^2 = 0. \end{cases} \quad (23)$$

Here, Z_t^2 represents the observable process when we have the $v_{opt}^2(t)$. We know when to find the MLE, we will find the root of the equation

$$F(Z_t^2, \vartheta) = \int_0^T X'_\vartheta(t, \vartheta) dZ_t^2 - \int_0^T X(t, \vartheta) X'_\vartheta(t, \vartheta) d\langle N \rangle_t = 0$$

where $X(t, \vartheta) = \lambda \ell(t)^* \zeta^2(t)$ and $X'_\vartheta(t, \vartheta)$ is the partial derivative of $X(t, \vartheta)$ with respect to ϑ .

The general Newton iteration method for the solution can be described by

$$\vartheta_{n+1} = \vartheta_n - \frac{F(\vartheta_n)}{F'(\vartheta_n)}.$$

In fact

$$F'(Z_t^2, \vartheta) = \int_0^T X''_{\vartheta\vartheta}(t, \vartheta) dZ_t^2 - \int_0^T |X'_\vartheta(t, \vartheta)|^2 d\langle N \rangle_t - \int_0^T X(t, \vartheta) X''_{\vartheta\vartheta} d\langle N \rangle_t.$$

Or when we develop dZ_t^2 , we can get that

$$F'(Z_t^2, \vartheta) = - \int_0^T |X'_\vartheta(t, \vartheta)|^2 d\langle N \rangle_t + \int_0^T X''_{\vartheta\vartheta} dN_t.$$

The second term is often negligible compared to the first one. By dropping it and making the first Newton iteration, we get an estimator $\hat{\vartheta}_T^2$ from an initial estimator $\bar{\vartheta}$ of the parameter ϑ :

$$\hat{\vartheta}_T^2 = \bar{\vartheta} + \frac{\int_0^T X'_\vartheta(t, \bar{\vartheta}) dZ_t^2 - \int_0^T X(t, \bar{\vartheta}) X'_\vartheta(t, \bar{\vartheta}) d\langle N \rangle_t}{\mathcal{I}(\bar{\vartheta})}$$

where $\mathcal{I}(\bar{\vartheta}) = \mathcal{I}_0^T(\bar{\vartheta})$ is the Fisher information of system (23). The difficulty is that the estimator depends on the observation time τ , the function $X'_\vartheta(t, \bar{\vartheta})$ is not nonanticipating, the integral $\int_0^\tau X'_\vartheta(t, \bar{\vartheta}) dZ_t^2$, so we can define the estimator as

$$\hat{\vartheta}_T^2 = \bar{\vartheta} + \frac{\int_\tau^T X'_\vartheta(t, \bar{\vartheta}) dZ_t^2 - \int_\tau^T X(t, \bar{\vartheta}) X'_\vartheta(t, \bar{\vartheta}) d\langle N \rangle_t}{\mathcal{I}_\tau^T(\bar{\vartheta})} \quad (24)$$

or we can write as

$$\sqrt{\mathcal{I}_\tau^T(\bar{\vartheta})}(\hat{\vartheta}_T - \vartheta) = \frac{1}{\sqrt{\mathcal{I}_\tau^T(\bar{\vartheta})}} \int_\tau^T X'_\vartheta(t, \bar{\vartheta}) dN_t + R \quad (25)$$

where the remainder

$$R = \frac{1}{\sqrt{\mathcal{I}_\tau^T(\bar{\vartheta})}} \int_\tau^T X'_\vartheta(t, \bar{\vartheta}) \left[X(t, \vartheta) - X(t, \bar{\vartheta}) - X'_\vartheta(t, \bar{\vartheta})(\vartheta - \bar{\vartheta}) \right] d\langle N \rangle_t \quad (26)$$

with Taylor formula, we can write it as

$$R = \frac{1}{\sqrt{\mathcal{I}_\tau^T(\bar{\vartheta})}} \int_\tau^T X'_\vartheta(t, \bar{\vartheta}) X''_{\vartheta\vartheta}(t, \bar{\vartheta}) d\langle N \rangle_t (\vartheta - \bar{\vartheta})^2. \quad (27)$$

where $\bar{\bar{\vartheta}}$ is a point between ϑ and $\bar{\vartheta}$. In view of equation (25), we know that, if there is no the remainder R , we can get the asymptotic efficiency of the estimator $\hat{\vartheta}_T^2$ since $\mathcal{I}_\tau^T(\bar{\vartheta})$ is asymptotically equivalent to $\mathcal{I}_T^2(\vartheta, v_{opt}^2(t))$ which is the Fisher information of the system (23). So we need the remainder is small. To study the remainder, we need to study the estimator $\bar{\vartheta}$.

3.5.2 Small Interval Estimator

We will observe the small interval $[0, \tau]$. Let us define a function $\bar{v}(t) = \rho\sqrt{2\lambda}t^{H-\frac{1}{2}}$ where ρ is a constant depending on τ . Assume that we are given a linear system

$$\begin{cases} \frac{d\zeta^3(t)}{d\langle N \rangle_t} = \lambda A_0 \otimes \mathbf{A}(t)\zeta^3(t) + b(t)\bar{v}(t), & \zeta^3(0) = 0, \\ dZ_t^3 = \lambda \ell(t)^* \zeta^3(t) d\langle N \rangle_t + dN_t, & Z_0^3 = 0. \end{cases} \quad (28)$$

where Z_t^3 is the observable process and we only observe the interval $[0, \tau]$ and get the estimator which we define as $\bar{\vartheta}$. We have the following Lemma:

Lemma 3.3. *when give the system (28), the MLE $\bar{\vartheta}$ for the parameter $\vartheta \in \mathbb{K}$ is asymptotically efficient provided that when $\tau \rightarrow 0$, $\tau^9 \rho^2 \rightarrow \infty$.*

The proof follows from the Ibragimov-Khasminskii Theorem. we know that the Fisher information is

$$\mathcal{I}_\tau(\vartheta) = \frac{1}{4} \int_0^\tau \left| \rho \frac{\partial}{\partial \vartheta} g(\vartheta, t) \right|^2 dt$$

where $g(\vartheta, t)$ is defined in (21). With Taylor's development of $g(\vartheta, t)$ and the condition $\tau^9 \rho^2 \rightarrow \infty$, we can verify the four conditions of Ibragimov-Khasminskii Theorem when $\tau \rightarrow 0$. So this Lemma follows.

In fact, we can have a more advanced result:

Corollary 1. *If we choose an arbitrary small interval $[0, \tau]$, ($\tau = o(1)$) as $T \rightarrow \infty$, and*

$$\int_0^\tau |\bar{v}(t)|^2 d\langle N \rangle_t = o(T)$$

we can obtain the estimator with the precision of order $\frac{1}{\sqrt{T}}$. More precisely, if $f(T) = o(\sqrt{T})$, we can find the estimator $\bar{\vartheta}$ such that $\bar{\vartheta} - \vartheta = O\left(\frac{1}{f(T)}\right)$.

3.5.3 Long Time Estimation

Now, we will return to the system (23), but using the estimator $\bar{\vartheta}$. we define a new function $\underline{v}(t)$ to replace $v_{opt}^2(\bar{\vartheta}, t)$, that is

$$\underline{v}(t) = \sqrt{2\lambda} t^{H-\frac{1}{2}} e^{\pm i \sqrt{\bar{\vartheta} - \frac{k^2}{2}}}$$

and get the MLE defined in (25). We have the theorem

Theorem 5. *When given the system (23), we have an asymptotically efficient two-stage estimator of parameter ϑ . The first is given in Corollary 1 (where, e.g. $\tau = T^{-\varepsilon}$, $\rho = \sqrt{T}$, ε is small positive). The second stage is $\hat{\vartheta}_T^2$ defined in (25).*

4 Proof of Lemma 3.1 and Lemma 3.2

In this section, we will prove Lemma 3.1 and Lemma 3.2. First of all, we will use Laplace transform to get the upper bound of the operator K_T . Then we will get the lower bound using the special value of the function $v(t)$.

4.1 Laplace Transform Proof Of Upper Bound

Let us introduce the pair process $\xi = ((\xi_1, \xi_2), 0 \leq t \leq T)$ with

$$\xi_t^1 = \left(\int_t^T \sigma^{\frac{1}{2}-H} \ell(\sigma)^* \varphi(\sigma) * dW_\sigma \right) \varphi^{-1}(t) \quad (29)$$

and

$$\xi_t^2 = \frac{\partial}{\partial \vartheta} \xi_t^1 \quad (30)$$

where W is a Winer process and $*dW_\sigma$ denotes the Itô backward integral (see [18]). It is worth emphasizing that

$$K_T(s, \sigma) = \frac{1}{4} \mathbf{E} \left(\xi_s^2 b(s) s^{\frac{1}{2}-H} \xi_\sigma^2 b(\sigma) \sigma^{\frac{1}{2}-H} \right) = \mathbf{E}(\mathcal{X}_\sigma \mathcal{X}_s),$$

where \mathcal{X} is the centered Gaussian process defined by :

$$\mathcal{X}_t = \frac{1}{2} \xi_t^2 b(t) t^{\frac{1}{2}-H}.$$

The process ξ also satisfies the following dynamic

$$-d\xi_t = \xi_t \mathcal{A}(t) d\langle N \rangle_t + \mathcal{L}(t) * dM_t, \quad \xi_T = 0,$$

with $M = (M_t, t \geq 0)$ a martingale of the same variance function as $N = (N_t, t \geq 0)$,

$$\mathcal{A}(t) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\vartheta & -k & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\vartheta & -k \end{pmatrix} \otimes \lambda \mathbf{A}(t) \text{ and } \mathcal{L}(t) = \sqrt{2\lambda} \begin{pmatrix} \ell(t)^* & 0 \end{pmatrix}$$

In fact, the covariance operator K_T is a symmetrical compact operator, we should estimate the spectral gap(the first eigenvalue $\nu_1(T)$). This estimation is based on the Laplace transform computation. Let us compute, for sufficiently small negative $a < 0$ the Laplace transform of $\int_0^T \mathcal{X}_t^2 dt$:

$$\begin{aligned} L_T(a) &= \mathbf{E}_\vartheta \exp \left\{ -a \int_0^T \mathcal{X}_t^2 dt \right\} \\ &= \mathbf{E}_\vartheta \exp \left\{ -a \int_0^T \left[\frac{1}{2} \left(\frac{\partial}{\partial \vartheta} \xi_t^1 \right) b(t) t^{\frac{1}{2}-H} \right] dt \right\} \end{aligned}$$

On the one hand, for $a > -\frac{1}{\nu_1(T)}$, since \mathcal{X} is a centered Gaussian process with covariance operator K_T , using Mercer's theorem and Parseval's inequality, $L_T(a)$ can be represented as:

$$L_T(a) = \prod_{i \geq 1} (1 + 2a\nu_i(T))^{-\frac{1}{2}}, \quad (31)$$

where $\nu_i(T)$, $i \geq 1$ is the sequence of positive eigenvalues of the covariance operator. On the other hand,

$$\begin{aligned} L_T(a) &= \mathbf{E}_\vartheta \exp \left\{ -\frac{a\lambda}{2} \int_0^T \xi_t \mathcal{M} \xi_t^* d\langle N \rangle_t \right\} \\ &= \exp \left\{ \frac{1}{2} \text{trace}(\mathcal{H}(t) \mathcal{L}(t)^* \mathcal{L}(t)) d\langle N \rangle_t \right\}, \end{aligned}$$

where

$$\mathcal{M}(t) = \begin{pmatrix} 0 & 0 \\ 0 & b(t)b(t)^* \end{pmatrix}$$

and $\mathcal{H}(t)$, $t \geq 0$ is the solution of Ricatti differential equation:

$$\frac{d\mathcal{H}(t)}{d\langle N \rangle_t} = \mathcal{H}(t)\mathcal{A}(t)^* + \mathcal{A}(t)\mathcal{H}(t) + \mathcal{H}(t)\mathcal{L}(t)^*\mathcal{L}(t)\mathcal{H}(t) - a\lambda\mathcal{M}(t), \quad (32)$$

with initial value $\mathcal{H}(0) = 0$, provided that the solution of equation (32) exists for any $0 \leq t \leq T$. It is well known that if $\det \Psi_1(t) > 0$, for any $t \in [0, T]$, then the solution \mathcal{H} of equation (32) can be written as $\mathcal{H}(t) = \Psi_1^{-1}(t)\Psi_2(t)$, where the pair of 8×8 matrices (Ψ_1, Ψ_2) satisfies the system of linear differential equation:

$$\frac{d\Psi_1(t)}{d\langle N \rangle_t} = -\Psi_1(t)\mathcal{A}(t) - \Psi_2(t)\mathcal{L}(t)^*\mathcal{L}(t), \quad \Psi_1(0) = \mathbf{Id}_{8 \times 8}, \quad (33)$$

$$\frac{d\Psi_2(t)}{d\langle N \rangle_t} = a\lambda\Psi_1(t)\mathcal{M}(t) + \Psi_2(t)\mathcal{A}(t)^*, \quad \Psi_2(0) = \mathbf{0}.$$

Moreover, under the condition $\det \Psi_1(t) > 0$, for any $t \in [0, T]$, the following equality holds:

$$\begin{aligned} L_T(a) &= \exp \left\{ -\frac{1}{2} \int_0^T \text{trace} \mathcal{A}(t) d\langle N \rangle_t \right\} (\det \Psi_1(T))^{-\frac{1}{2}} \\ &= \exp\{kT\} (\det \Psi_1(T))^{-\frac{1}{2}}, \end{aligned}$$

or equivalently using (31),

$$\prod_{i \geq 1} (1 + 2a\nu_i(T)) = \exp\{-2kT\} (\det \Psi_1(T)). \quad (34)$$

Let us note here that the solution of linear system (33) exist for any $t > 0$ and for any $a \in \mathbb{C}$. For $a = 0$, $\det \Psi_1(t) = \exp\{2kt\} > 0$. Due to the continuity property of the solutions of linear differential equations with respect to a parameter, for all $T > 0$, there exists $a(T) < 0$ such that

$$\inf_{t \in [0, T]} \det \Psi_1(t) > 0.$$

Therefore, equality (34) holds in an open set in \mathbb{C} , containing 0. Compactness of the covariance operator implies due to the Weierstrass theorem, the

analytic property of $\prod_{i \geq 1} (1 + 2a\nu_i(T))$ with respect to a . Hence, equality (34) holds for any $a \in \mathbb{C}$.

Now, we rewrite the system of (Ψ_1, Ψ_2) that

$$\frac{d(\Psi_1(t), \Psi_2(t)\mathbf{J})}{d\langle N \rangle_t} = (\Psi_1(t), \Psi_2(t)\mathbf{J}) \cdot (\Upsilon \otimes \lambda\mathbf{A}(t)) \quad (35)$$

where $\mathbf{J} = \begin{pmatrix} J & J & J & J \\ J & J & J & J \\ J & J & J & J \\ J & J & J & J \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and

$$\Upsilon = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \vartheta & k & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \vartheta & k & 0 & 0 & 0 & -a \\ -2 & 0 & 0 & 0 & 0 & -\vartheta & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -k & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -\vartheta \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -k \end{pmatrix}$$

The eigenfunction of Υ is that

$$(y^2 - ky + \vartheta)^2(y^2 + ky + \vartheta)^2 + 2a = 0$$

4.1.1 The Case Of $k^2 \geq 2\vartheta$

In this case, when $-\frac{\vartheta^4}{2} < a < 0$, let $(y_i)_{i=1,\dots,8}$ be the eigenvalues of the matrix Υ , it can be checked that

$$\det \Psi_1(T) = \exp((y_1 + y_3 + y_5 + y_7)T) (C + O(\frac{1}{T})).$$

where C is a constant and there are 3 cases with different y_i .

(1) $k^2 \geq 4\vartheta$, **there are 8 real eigenvalues** we get that

$$y_1 = \sqrt{\frac{k^2 - 2\vartheta + \sqrt{k^4 - 4k^2\vartheta + 4\sqrt{-2a}}}{2}},$$

$$y_3 = \sqrt{\frac{k^2 - 2\vartheta - \sqrt{k^4 - 4k^2\vartheta + 4\sqrt{-2a}}}{2}},$$

$$y_5 = \sqrt{\frac{k^2 - 2\vartheta + \sqrt{k^4 - 4k^2\vartheta - 4\sqrt{-2a}}}{2}},$$

$$y_7 = \sqrt{\frac{k^2 - 2\vartheta - \sqrt{k^4 - 4k^2\vartheta - 4\sqrt{-2a}}}{2}}.$$

(2) $k^2 \geq 4\vartheta$ or $2\vartheta \leq k^2 \leq 4\vartheta$, there are 4 real eigenvalues and 4 complex eigenvalues.

$$y_1 = \sqrt{\frac{k^2 - 2\vartheta + \sqrt{k^4 - 4k^2\vartheta + 4\sqrt{-2a}}}{2}},$$

$$y_3 = \sqrt{\frac{k^2 - 2\vartheta - \sqrt{k^4 - 4k^2\vartheta + 4\sqrt{-2a}}}{2}},$$

$y_5 + y_7 = 2\sqrt{m}$ where m and n are the solutions of the equation $m^2 - n^2 = k^2 - 2\vartheta$ and $2mn = \sqrt{4\sqrt{-2a} - k^4 + 4k^2\vartheta}$.

(3) $2\vartheta \leq k^2 \leq 4\vartheta$, there are 8 complex eigenvalues $y_1 + y_3 = 2\sqrt{p}$ where p and q are the solutions of the equation $p^2 - q^2 = k^2 - 2\vartheta$ and $2pq = \sqrt{-4\sqrt{-2a} - k^4 + 4k^2\vartheta}$. $y_5 + y_7 = 2\sqrt{m}$ where m and n are the solutions of the equation $m^2 - n^2 = k^2 - 2\vartheta$ and $2mn = \sqrt{4\sqrt{-2a} - k^4 + 4k^2\vartheta}$.

Therefor, due to the equality (34), we have that when $k^2 \geq 2\vartheta$, $\prod_{i \geq 1} (1 + 2a\nu_i(T)) > 0$ for any $a > -\frac{\vartheta^4}{2}$. It means that

$$\nu_1(T) \leq \frac{1}{\vartheta^4}$$

4.1.2 The Case Of $k^2 < 2\vartheta$

Now let us consider $k^2 < 2\vartheta$, when $-\frac{(k^4-4k^2\vartheta)^2}{32} < a < 0$, there are 8 complex eigenvalues and it can be check that

$$\det \Psi_1(T) = \exp((y_1 + y_3 + y_5 + y_7)T) (C + O(\frac{1}{T})).$$

where $y_1 + y_3 = 2\sqrt{p}$, p and q are the solutions of the equation $p^2 - q^2 = k^2 - 2\vartheta$ and $2pq = \sqrt{-4\sqrt{-2a} - k^4 + 4k^2\vartheta}$. $y_5 + y_7 = 2\sqrt{m}$, m and n are the solutions of the equation $m^2 - n^2 = k^2 - 2\vartheta$ and $2mn = \sqrt{4\sqrt{-2a} - k^4 + 4k^2\vartheta}$.

Therefore, with the equality (34), $\prod_{i \geq 1} (1 + 2a\nu_i(T)) > 0$ for any $a > -\frac{(k^4-4k^2\vartheta)^2}{32}$ which means that

$$\nu_1(T) \leq \frac{16}{(k^4 - 4k^2\vartheta)^2}$$

4.2 Lower Bound of The Operator

First of all, let us consider the same problem with standard Brownian motion. Let us consider the model

$$\begin{cases} \frac{d^2x}{dt^2} + k\frac{dx}{dt} + \vartheta x = u(t), & x(0) = 0, \\ dY_t = x(t)dt + dW_t, & Y_0 = 0. \end{cases} \quad (36)$$

where W_t is a standard Brownian motion. We can get the Fisher information of this system

$$\mathcal{I}_T(\vartheta, u) = \int_0^T \left(\frac{\partial x(t)}{\partial \vartheta} \right)^2 dt. \quad (37)$$

Or we can write as

$$\mathcal{I}_T(\vartheta, u) = \int_0^T \left[\begin{pmatrix} 1 & 0 \end{pmatrix} \frac{\partial X(t)}{\partial \vartheta} \right]^2 dt \quad (38)$$

where $X(t)$ is the solution of the equation

$$dX(t) = A_0 X(t)dt + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t)dt.$$

A_0 is defined in (9)

With some computation (see [5, 24]), we can get that when $v(t) = v_{opt}^1(t)$ and $v(t) = v_{opt}^2(t)$, the Fisher information $\mathcal{I}_T(\vartheta, v)$ in (14) is equal to $\mathcal{I}_T(\vartheta, u)$ in (38) when $u(t) = 1$ and $u(t) = e^{i\omega t}$ where $\omega = \pm\sqrt{\vartheta - \frac{k^2}{2}}$, That is to say we can calculate only the $\mathcal{I}_T(\vartheta, u)$ in (37) which gives us that

$$\frac{\mathcal{I}_T(\vartheta, v_{opt}^1(t))}{T} = \frac{1}{\vartheta^4}.$$

and

$$\frac{\mathcal{I}_T(\vartheta, v_{opt}^2(t))}{T} = \frac{16}{(k^4 - 4k^2\vartheta)^2}.$$

which achieves the proof.

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